

arXiv:hep-th/0506253v3 2 Oct 2006

TOWARDS AN ASHTEKAR FORMALISM IN TWELVE DIMENSIONS¹

J. A. Nieto²

*Facultad de Ciencias Físico-Matemáticas de la Universidad Autónoma
de Sinaloa, 80010, Culiacán Sinaloa, México*

Abstract

We discuss the Ashtekar formalism from the point of view of twelve dimensions. We first focus on the $2 + 10$ spacetime signature and then we consider the transition $2 + 10 \rightarrow (2 + 2) + (0 + 8)$. We argue that both sectors $2 + 2$ and $0 + 8$, which are exceptional signatures, can be analyzed from the point of view of a self-dual action associated with the Ashtekar formalism.

Keywords: Ashtekar theory, twelve dimensions, two time physics.

Pacs numbers: 04.60.-m, 04.65.+e, 11.15.-q, 11.30.Ly

October, 2006

¹Dedicated to Octavio Obregón on the occasion of his sixtieth birthday

²nieto@uas.uasnet.mx

Over the years it has become clear that considering gravitational and gauge theories with more than one time coordinate is a very interesting and useful idea for understanding different aspects of traditional gravitational and gauge field theories with only one time (see Ref. [1] and references therein). In particular, the $2+10$ -dimensional spacetime signature has emerged as an interesting possibility for the understanding of both supergravity and super Yang-Mills theory in $D = 11$ (see Ref. [2]). Thus, by seriously taking a $2+10$ -dimensional gravity one may be interested in various possibilities offered by this theory. For instance, one may be interested in a realistic theory in four dimensions via the symmetry braking $2+10 \rightarrow (1+3) + (1+7)$. However, this may not be the only attractive possibility. In fact, one may think on the alternative transition

$$2+10 \rightarrow (2+2) + (0+8). \quad (1)$$

Of course, in this case one should not have a direct connection with our four dimensional real World. Nevertheless, the signature $2+2$ has been considered in connection with a world volume of $2+2$ -brane (see [3]). In fact, the $2+2$ -brane arises in $N = 2$ theories which require two times for its complete formulation [4]. Another source of interest in the $2+2$ signature is that such a signature admits a Majorana-Weyl spinors and self-dual gauge fields formulation [5]. Moreover, it has been shown that the symmetry $SL(2, R)$ makes the $2+2$ signature an exceptional one [6]. On the other hand, the signature $0+8$ is euclidean and in principle can be treated with the traditional methods such as the octonion algebraic approach [7]. In pass, it is interesting to observe that octonion algebra is also exceptional in the sense of Hurwitz theorem (see Ref. [8] and references therein). Thus, we see that both $2+2$ and $0+8$ are exceptional signatures and therefore these observations make the transition (1) worthwhile of being studied.

Here, we shall discuss the signatures $2+2$ and $0+8$ from the point of view of ‘self-dual’ actions associated with the Ashtekar formalism (see Ref. [9] and references therein). For that purpose let us assume that the spacetime manifold M^{2+10} can be broken up into the form $M^{10+2} \rightarrow M^{2+2} \times M^{0+8}$. This implies that the $SO(2, 10)$ Lovelock type curvature (see Refs. [10]-[13] and references therein)

$$\mathcal{R}_{\hat{\mu}\hat{\nu}}^{\hat{A}\hat{B}} = R_{\hat{\mu}\hat{\nu}}^{\hat{A}\hat{B}} + \Sigma_{\hat{\mu}\hat{\nu}}^{\hat{A}\hat{B}}, \quad (2)$$

with

$$R_{\hat{\mu}\hat{\nu}}^{\hat{A}\hat{B}} = \partial_{\hat{\mu}}\omega_{\hat{\nu}}^{\hat{A}\hat{B}} - \partial_{\hat{\nu}}\omega_{\hat{\mu}}^{\hat{A}\hat{B}} + \omega_{\hat{\mu}}^{\hat{A}\hat{C}}\omega_{\hat{\nu}\hat{C}}^{\hat{B}} - \omega_{\hat{\mu}}^{\hat{B}\hat{C}}\omega_{\hat{\nu}\hat{C}}^{\hat{A}} \quad (3)$$

and

$$\Sigma_{\hat{\mu}\hat{\nu}}^{\hat{A}\hat{B}} = e_{\hat{\mu}}^{\hat{A}} e_{\hat{\nu}}^{\hat{B}} - e_{\hat{\nu}}^{\hat{A}} e_{\hat{\mu}}^{\hat{B}}, \quad (4)$$

can be split into the form

$$\mathcal{R}_{ij}^{AB} = R_{ij}^{AB} + \Sigma_{ij}^{AB} \quad (5)$$

and

$$\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = R_{\mu\nu}^{\hat{a}\hat{b}} + \Sigma_{\mu\nu}^{\hat{a}\hat{b}}, \quad (6)$$

with the corresponding definitions (3) and (4) for R_{ij}^{AB} , Σ_{ij}^{AB} , $R_{\mu\nu}^{\hat{a}\hat{b}}$ and $\Sigma_{\mu\nu}^{\hat{a}\hat{b}}$. In addition, we assume that \mathcal{R}_{ij}^{AB} and $\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}$ ‘live’ in M^{2+2} and M^{0+8} , with $SO(2, 2)$ and $SO(8)$ as the corresponding gauge groups, respectively.

Let us now consider a MacDowell-Mansouri type action [11]-[13] in a $2 + 10$ -dimensional spacetime [14]

$$\mathcal{S} = \int_{M^{2+10}} \Omega^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}} \mathcal{R}_{\hat{\mu}\hat{\nu}}^{\hat{A}\hat{B}} \mathcal{R}_{\hat{\alpha}\hat{\beta}}^{\hat{C}\hat{D}} \Omega_{\hat{A}\hat{B}\hat{C}\hat{D}}, \quad (7)$$

where $\Omega^{\hat{\mu}\hat{\nu}\hat{\alpha}\hat{\beta}}$ is a completely antisymmetric constant in M^{10+2} and Ω_{ABCD} is also a completely antisymmetric constant associated with the $SO(2, 10)$ group, yet to be chosen. Assuming the transition $M^{10+2} \rightarrow M^{2+2} \times M^{0+8}$ we find that the action (7) may be split as

$$\mathcal{S} = \int_{M^{2+2}} \varepsilon^{ijkl} \mathcal{R}_{ij}^{AB} \mathcal{R}_{kl}^{CD} \varepsilon_{ABCD} + \int_{M^{0+8}} \eta^{\mu\nu\alpha\beta} \mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} \mathcal{R}_{\alpha\beta}^{\hat{c}\hat{d}} \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}. \quad (8)$$

Here, ε^{ijkl} and ε_{ABCD} are completely antisymmetric objects linked to the signature $2+2$, while $\eta^{\mu\nu\alpha\beta}$ and $\eta_{\hat{a}\hat{b}\hat{c}\hat{d}}$ are completely antisymmetric objects linked to the signature $0+8$. The next step in our quest of associating an Ashtekar formalism with the spacetimes of signatures $2+2$ and $0+8$ is to consider the self-dual (antiself-dual) sector of the action (8).

Let us first focus on the first term in (8);

$$S_{2+2} = \int_{M^{2+2}} \varepsilon^{ijkl} \mathcal{R}_{ij}^{AB} \mathcal{R}_{kl}^{CD} \varepsilon_{ABCD}. \quad (9)$$

We recognize this action as the MacDowell-Mansouri action for spacetime of signature $2+2$. Before we write the self dual sector of (9) it is convenient to discuss some of the properties of the object ε_{ABCD} . First, let us set $\varepsilon_{1234} = 1$. So we find that

$$\varepsilon^{ABCD} = \eta^{AE} \eta^{BF} \eta^{CG} \eta^{DH} \varepsilon_{EFGH}, \quad (10)$$

where $\eta_{AB} = \text{diag}(-1, -1, 1, 1)$, gives $\varepsilon^{1234} = 1$. We observe that $\text{Det}\eta_{AB} = 1$. Hence, we get

$$\varepsilon^{ABCD}\varepsilon_{EFCD} = 2\delta_{EF}^{AB}. \quad (11)$$

Here, we used the definition $\delta_{EF}^{AB} \equiv \delta_E^A \delta_F^B - \delta_E^B \delta_F^A$, where δ_E^A is the Kronecker delta. This means that the property (11) of ε_{ABCD} is exactly the same as the corresponding euclidean quantity.

Let us now define the dual curvature

$$*\mathcal{R}_{ij}^{AB} = \frac{1}{2}\varepsilon_{CD}^{AB}\mathcal{R}_{ij}^{CD}. \quad (12)$$

Using (11) we see that

$$**\mathcal{R}_{ij}^{AB} = \mathcal{R}_{ij}^{AB}. \quad (13)$$

In this way, the self-dual (antiself-dual) curvature

$$\pm\mathcal{R}_{ij}^{AB} = \frac{1}{2}(\mathcal{R}_{ij}^{AB} \pm *\mathcal{R}_{ij}^{AB}) \quad (14)$$

gives

$$*\pm\mathcal{R}_{ij}^{AB} = (\pm)\pm\mathcal{R}_{ij}^{AB}, \quad (15)$$

which means that $\pm\mathcal{R}_{ij}^{AB}$ is self dual (antiself-dual). In fact, we have

$$\mathcal{R}_{ij}^{AB} = +\mathcal{R}_{ij}^{AB} + -\mathcal{R}_{ij}^{AB}.$$

This implies that the action (see Refs. [13], [15], [16])

$$+\mathcal{S}_{2+2} = \int_{M^{2+2}} \varepsilon^{ijkl} \mathcal{R}_{ij}^{AB+} \mathcal{R}_{kl}^{CD} \varepsilon_{ABCD} \quad (16)$$

corresponds to the self-dual sector of the action (9). A similar action can be written for the antiself-dual sector of (9). In fact, we have $\mathcal{S}_{2+2} = +\mathcal{S}_{2+2} + -\mathcal{S}_{2+2}$.

Now we would like to discuss the consequences of (16). For this purpose we first write (14) in the form

$$\pm\mathcal{R}_{ij}^{AB} = \frac{1}{2} \pm B_{KL}^{AB} \mathcal{R}_{ij}^{KL}, \quad (17)$$

where

$$\pm B_{KL}^{AB} = \frac{1}{2}(\delta_{KL}^{AB} \pm \varepsilon_{KL}^{AB}). \quad (18)$$

By straightforward computation one finds that the projector $\pm B_{KL}^{AB}$ satisfies the property

$$\pm B_{MN}^{AB\pm} B_{RS}^{CD} \varepsilon_{ABCD} = \pm 4^\pm B_{MNRs}. \quad (19)$$

Therefore, using (19) we see that (16) can also be written as

$$^+S_{2+2} = \int_{M^{2+2}} \varepsilon^{ijkl} \mathcal{R}_{ij}^{AB} \mathcal{R}_{kl}^{CD} B_{ABCD}. \quad (20)$$

Using (18) we discover that (20) results in

$$^+S_{2+2} = \frac{1}{2} \int_{M^{2+2}} \varepsilon^{ijkl} \mathcal{R}_{ij}^{AB} \mathcal{R}_{kl}^{CD} \varepsilon_{ABCD} + \frac{1}{2} \int_{M^{2+2}} \varepsilon^{ijkl} \mathcal{R}_{ij}^{AB} \mathcal{R}_{kl}^{CD} \eta_{ABCD}, \quad (21)$$

where $\eta_{ABCD} \equiv \eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}$. The first term in (21) can be split using (5) in the Euler topological invariant and the Einstein-Hilbert action with cosmological constant, while the second term leads to the Pontrjagin topological invariant. Therefore, up to topological invariants the action (16) is equivalent to the Einstein-Hilbert action with cosmological constant. It is worth mentioning how the cosmological constant arises from the first term of (21). Since $\mathcal{R}_{ij}^{AB} = R_{ij}^{AB} + \Sigma_{ij}^{AB}$ and $\Sigma_{ij}^{AB} = e_i^A e_j^B - e_i^B e_j^A$ we observe that under the rescaling $e_i^A \rightarrow \lambda e_i^A$ we shall get the transformations $\varepsilon^{ijkl} \Sigma_{ij}^{AB} R_{kl}^{CD} \varepsilon_{ABCD} \rightarrow \lambda^2 \varepsilon^{ijkl} \Sigma_{ij}^{AB} R_{kl}^{CD} \varepsilon_{ABCD}$ and $\varepsilon^{ijkl} \Sigma_{ij}^{AB} \Sigma_{kl}^{CD} \varepsilon_{ABCD} \rightarrow \lambda^4 \varepsilon^{ijkl} \Sigma_{ij}^{AB} \Sigma_{kl}^{CD} \varepsilon_{ABCD}$ which means that under the rescaling $^+S_{2+2} \rightarrow \frac{1}{\lambda^2} ^+S_{2+2}$ one can identify, up to numerical factor, λ^2 with the cosmological constant (see Ref. [12] for details).

Let us now consider the second term in (8) (see Ref. [14])

$$S_{0+8} = \int_{M^{0+8}} \eta^{\mu\nu\alpha\beta} \mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} \mathcal{R}_{\alpha\beta}^{\hat{c}\hat{d}} \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}. \quad (22)$$

First we need to clarify the meaning of the completely antisymmetric objects $\eta_{\hat{a}\hat{b}\hat{c}\hat{d}}$ (or $\eta^{\mu\nu\alpha\beta}$). The key idea is to relate $\eta_{\hat{a}\hat{b}\hat{c}\hat{d}}$ to the octonion structure constants $C_{\hat{a}\hat{b}}^{\hat{c}}$ in the form

$$\eta_{8abc} \equiv \varsigma C_{abc} \quad (23)$$

and

$$\eta_{abcd} \equiv \frac{1}{3!} \varepsilon_{abcdefg} C^{efg}, \quad (24)$$

where the indices a, b, \dots etc run from 1 to 7, $\varepsilon_{abcdefg}$ is the completely antisymmetric symbol in seven dimensions and $\varsigma = \pm$. Using (23) and (24) it can be shown that $\eta_{\hat{a}\hat{b}\hat{c}\hat{d}}$ is self-dual:

$$\eta_{\hat{a}\hat{b}\hat{c}\hat{d}} = \frac{\varsigma}{4!} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}\hat{g}\hat{h}} \eta^{\hat{e}\hat{f}\hat{g}\hat{h}}. \quad (25)$$

For $\varsigma = 1$, it is self-dual (and for $\varsigma = -1$ is antiself-dual). One can verify that four-rank completely antisymmetric tensor $\eta_{\hat{a}\hat{b}\hat{c}\hat{d}}$ (also $\eta_{\mu\nu\alpha\beta}$) satisfies the relations [17]-[19] (see also Refs. [7] and [20]),

$$\eta_{\hat{a}\hat{b}\hat{c}\hat{d}} \eta^{\hat{e}\hat{f}\hat{c}\hat{d}} = 6\delta_{\hat{a}\hat{b}}^{\hat{e}\hat{f}} + 4\eta_{\hat{a}\hat{b}}^{\hat{e}\hat{f}}, \quad (26)$$

$$\eta_{\hat{a}\hat{b}\hat{c}\hat{d}} \eta^{\hat{e}\hat{b}\hat{c}\hat{d}} = 42\delta_{\hat{a}}^{\hat{e}}, \quad (27)$$

$$\eta_{\hat{a}\hat{b}\hat{c}\hat{d}} \eta^{\hat{a}\hat{b}\hat{c}\hat{d}} = 336. \quad (28)$$

The next step is to introduce the dual of $\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}$ in the form

$$*\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = \frac{1}{2} \eta_{\hat{c}\hat{d}}^{\hat{a}\hat{b}} \mathcal{R}_{\mu\nu}^{\hat{c}\hat{d}}. \quad (29)$$

The self-dual and antiself-dual parts $^{\pm}\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}$ of $\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}$ are defined as

$$^+\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = \frac{1}{4} (\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} + *\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}) \quad (30)$$

and

$$^-\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = \frac{1}{4} (3\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} - *\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}), \quad (31)$$

respectively. Since

$$**\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = 3\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} + 2*\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}, \quad (32)$$

we see that

$$*^+\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = 3^+\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} \quad (33)$$

and

$$*^-\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = -^-\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}. \quad (34)$$

Thus, up to a numerical factor we see that $^+\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}$ and $^-\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}$ play, in fact, the role of the self-dual and antiself-dual parts, respectively of $\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}}$. It turns out to be convenient to write (30) as

$${}^+\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = \frac{1}{2} {}^+\Lambda_{\hat{c}\hat{d}}^{\hat{a}\hat{b}} \mathcal{R}_{\mu\nu}^{\hat{c}\hat{d}}, \quad (35)$$

where

$${}^+\Lambda_{\hat{c}\hat{d}}^{\hat{a}\hat{b}} = \frac{1}{4} (\delta_{\hat{c}\hat{d}}^{\hat{a}\hat{b}} + \eta_{\hat{c}\hat{d}}^{\hat{a}\hat{b}}). \quad (36)$$

While, (31) can be written in the form

$${}^-\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = \frac{1}{2} {}^-\Lambda_{\hat{c}\hat{d}}^{\hat{a}\hat{b}} \mathcal{R}_{\mu\nu}^{\hat{c}\hat{d}}, \quad (37)$$

with

$${}^-\Lambda_{\hat{c}\hat{d}}^{\hat{a}\hat{b}} = \frac{1}{4} (3\delta_{\hat{c}\hat{d}}^{\hat{a}\hat{b}} - \eta_{\hat{c}\hat{d}}^{\hat{a}\hat{b}}). \quad (38)$$

Now, we would like to propose the action

$$\begin{aligned} {}^\pm \mathcal{S}_{0+8} = & \frac{1}{{}^\pm \tau} \int_{M^{0+8}} \eta^{\mu\nu\alpha\beta} {}^\pm \mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} \mathcal{R}_{\alpha\beta}^{\hat{c}\hat{d}} \eta_{\hat{a}\hat{b}\hat{c}\hat{d}} + \frac{1}{{}^\mp \tau} \int_{M^{0+8}} \eta^{\mu\nu\alpha\beta} {}^\mp \mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} \mathcal{R}_{\alpha\beta}^{\hat{c}\hat{d}} \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}, \end{aligned} \quad (39)$$

which is a generalization of the action (22). Here, ${}^+\tau$ and ${}^-\tau$ are two constant parameters. It is worth mentioning that the proposal (39) emerged from the observation that ${}^\pm \Lambda$ are projection operators. In fact, one can prove that the objects ${}^+\Lambda$ and ${}^-\Lambda$, given in (36) and (38) respectively, satisfy [19]

$${}^+\Lambda + {}^-\Lambda = 1, \quad (40)$$

$${}^+\Lambda {}^-\Lambda = {}^-\Lambda {}^+\Lambda = 0, \quad (41)$$

$${}^+\Lambda^2 = {}^+\Lambda, \quad (42)$$

and

$${}^-\Lambda^2 = {}^-\Lambda. \quad (43)$$

Here, ${}^\pm \Lambda^2$ mean $\frac{1}{4} {}^\pm \Lambda_{\hat{c}\hat{d}}^{\hat{a}\hat{b}} \Lambda_{\hat{g}\hat{h}}^{\hat{e}\hat{f}} \delta_{\hat{a}\hat{b}\hat{e}\hat{f}}$.

Let us focus on the self-dual part of (39):

$${}^+\mathcal{S}_{0+8} = \frac{1}{{}^+\tau} \int_{M^{0+8}} \eta^{\mu\nu\alpha\beta} {}^+\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} \mathcal{R}_{\alpha\beta}^{\hat{c}\hat{d}} \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}. \quad (44)$$

Presumably, most of the computations that we shall develop below in connection with ${}^+\mathcal{S}_{0+8}$ may also be applied to the antiself-dual sector ${}^-\mathcal{S}_{0+8}$. It is worth mentioning that the action (39) is the analogue of the action proposed by Nieto [14] in eight dimensions with signature $1+7$. Let us start observing that since

$${}^+\mathcal{R}_{\mu\nu}^{\hat{a}\hat{b}} = {}^+R_{\mu\nu}^{\hat{a}\hat{b}} + {}^+\Sigma_{\mu\nu}^{\hat{a}\hat{b}}, \quad (45)$$

one finds that the action (44) becomes

$${}^+\mathcal{S}_{0+8} = \frac{1}{{}^+\tau} \int_{M^{0+8}} (T + K + C), \quad (46)$$

with

$$T = \eta^{\mu\nu\alpha\beta+} R_{\mu\nu}^{\hat{a}\hat{b}+} R_{\alpha\beta}^{\hat{c}\hat{d}} \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}, \quad (47)$$

$$K = 2\eta^{\mu\nu\alpha\beta+} \Sigma_{\mu\nu}^{\hat{a}\hat{b}+} R_{\alpha\beta}^{\hat{c}\hat{d}} \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}, \quad (48)$$

and

$$C = \eta^{\mu\nu\alpha\beta+} \Sigma_{\mu\nu}^{\hat{a}\hat{b}+} \Sigma_{\alpha\beta}^{\hat{c}\hat{d}} \eta_{\hat{a}\hat{b}\hat{c}\hat{d}}. \quad (49)$$

Using (35) and (36), it is not difficult to see that T can be identified with a topological invariant in eight dimensions analogous to Pontrjagin and Euler invariants in four dimensions. At this respect, it is worth mentioning that in the case of G_2 -invariant super Yang Mills theory [21] a topological term of the form

$$\mathcal{S}_{0+8} = \frac{1}{{}^+\tau} \int_{M^{0+8}} \eta^{\mu\nu\alpha\beta} F_{\mu\nu}^a F_{\alpha\beta}^b g_{ab}, \quad (50)$$

where $F_{\mu\nu}^a$ is the Yang-Mills field strength and g_{ab} is the group invariant metric, has been considered. Thus, the term T in (47) can be considered as the ‘gravitational’ analogue of (50). Similarly, K should lead to a kind of gravity in eight dimensions. Finally, C may be identified as the analogue of a cosmological constant term in the following sense. For a cosmological constant one would expect a term of the form

$$\frac{1}{8!} \int_{M^{0+8}} \varepsilon^{\mu_1 \dots \mu_8} \varepsilon_{\hat{a}_1 \dots \hat{a}_8} e_{\mu_1}^{\hat{a}_1} \dots e_{\mu_8}^{\hat{a}_8} = \int_{M^{0+8}} \det(e_{\mu}^{\hat{a}}). \quad (51)$$

But, C is quartic in $e_{\mu}^{\hat{a}}$ and then at first sight one may say that does not contain $\det(e_{\mu}^{\hat{a}})$. However, due to the self-dual relation (25) one may write C in the form

$$C = \frac{1}{4!} \eta^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{\hat{a}\hat{b}+} \Sigma_{\alpha\beta}^{\hat{c}\hat{d}} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}\hat{g}\hat{h}} \eta^{\hat{e}\hat{f}\hat{g}\hat{h}}. \quad (52)$$

Thus, using the identity $\varepsilon_{\hat{a}_1 \dots \hat{a}_8} e_{\mu_1}^{\hat{a}_1} \dots e_{\mu_8}^{\hat{a}_8} = \det(e_{\mu}^{\hat{a}}) \varepsilon_{\mu_1 \dots \mu_8}$ one may obtain that $C \sim \det(e_{\mu}^{\hat{a}})$. In fact, (52) can be rewritten as

$$\begin{aligned} C &= \frac{1}{4!} \eta^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{\tau\lambda} + \Sigma_{\alpha\beta}^{\sigma\rho} \varepsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}\hat{g}\hat{h}} e_{\tau}^{\hat{a}} e_{\lambda}^{\hat{b}} e_{\sigma}^{\hat{c}} e_{\rho}^{\hat{d}} e_{\gamma}^{\hat{e}} e_{\delta}^{\hat{f}} e_{\xi}^{\hat{g}} e_{\theta}^{\hat{h}} \tilde{\eta}^{\gamma\delta\xi\theta} \\ &= \det(e_{\mu}^{\hat{a}}) \frac{1}{4!} \eta^{\mu\nu\alpha\beta} + \Sigma_{\mu\nu}^{\tau\lambda} + \Sigma_{\alpha\beta}^{\sigma\rho} \varepsilon_{\tau\lambda\sigma\rho\gamma\delta\xi\theta} \tilde{\eta}^{\gamma\delta\xi\theta}, \end{aligned} \quad (53)$$

where $+\Sigma_{\mu\nu}^{\tau\lambda} e_{\tau}^{\hat{a}} e_{\lambda}^{\hat{b}} = +\Sigma_{\mu\nu}^{\hat{a}\hat{b}}$ and $e_{\gamma}^{\hat{e}} e_{\delta}^{\hat{f}} e_{\xi}^{\hat{g}} e_{\theta}^{\hat{h}} \tilde{\eta}^{\gamma\delta\xi\theta} = \eta^{\hat{e}\hat{f}\hat{g}\hat{h}}$. This shows that in principle we may have $C \sim \det(e_{\mu}^{\hat{a}})$. However, in the process we have introduced two new objects $+\Sigma_{\mu\nu}^{\tau\lambda}$ and $\tilde{\eta}^{\gamma\delta\xi\theta}$ which may lead to an alternative result when they are again written in terms of $+\Sigma_{\mu\nu}^{\hat{a}\hat{b}}$ and $\eta^{\hat{e}\hat{f}\hat{g}\hat{h}}$ respectively. Other possibility is to consider C as a new type of cosmological constant term not necessarily related to $\det(e_{\mu}^{\hat{a}})$. In this case it is necessary to consider that, in general, the ε -symbol is Lorentz invariant in any dimension, but in contrast the η -symbol is only $SO(7)$ -invariant in eight dimensions (see Ref. [19]). Therefore, the η -symbol spoils the Lorentz invariance of the action $+\mathcal{S}_{0+8}$ given in (50) and in particular of the C term, but maintains a hidden $SO(7)$ -invariance. In fact, this is a general phenomenon in field theories involving the η -symbol (see, for instance, Refs. [21] and [7]).

Let us summarize our results. We started with a $2+10$ dimensional gravitational theory and we assumed a possible symmetry braking of the form $2+10 \rightarrow (2+2) + (0+8)$. We proved that classically it makes sense to associate both signatures $2+2$ and $0+8$ with the Ashtekar formalism. Although our procedure was similar to the case $2+10 \rightarrow (1+3) + (1+7)$, the steps were necessary if eventually one desires to develop an Ashtekar canonical quantization for the signatures $2+2$ and $0+8$.

Since one of the most interesting candidates for the so-called M -theory is a theory of a $2+2$ -brane embedded in $2+10$ dimensional background target spacetime (see [3] and Refs. therein) our formalism points out a possible connection between Ashtekar formalism and M -theory. In fact, this version of M -theory evolved from the observation [4] that the complex structure of $N=2$ strings requires a target spacetime of signature $2+2$ rather than $1+9$ as the usual $N=1$ string theory. Thus, a natural step forward was to consider the $N=(2,1)$ heterotic string [22]. In this scenario, it was observed that a consistent $N=(2,1)$ string should consider right-movers ‘living’ in $2+2$ dimensions and left-movers in $2+10$ dimensions. The connection with our work comes from the fact that the dynamics of a $2+2$ -branes leads to self-dual

gravity coupled to self-dual supermatter in $2+10$ dimensions. And self-duality in this signatures is precisely what we have considered in this work.

Another reason for expecting a connection between M -theory and Ashtekar formalism comes from the link between oriented matroid theory [23] and two time physics. In fact, it has been proved [24] that oriented matroids may be related to M -theory by various routes [25]-[29], and in particular via two time physics [24]. Moreover, it has been proposed that oriented matroid theory may provide a mathematical framework for M -theory [29]-[30]. Thus, a connection between Ashtekar formalism and oriented matroid theory seems to be an interesting possibility.

The actions ${}^+\mathcal{S}_{2+2}$ and ${}^+\mathcal{S}_{0+8}$, given in (16) and (44) respectively, should in principle admit steps toward a canonical quantization similar to the steps given after the Jacobson-Smolin-Samuel action in four dimensions [31]-[32]. However, while canonical quantization in four dimensions of the Jacobson-Smolin-Samuel action leads to quantum states of the form $\exp(S_{cs})$, where S_{cs} is a Chern-Simons action, the quantum states in the signatures $2+2$ and $0+8$ could be very different and surprising. The main reason for this is that the signatures $2+2$ and $0+8$ are exceptional, and therefore, one should expect that the corresponding canonical quantizations are also exceptional.

We should mention something about the transition $M^{10+2} \rightarrow M^{2+2} \times M^{0+8}$ which allows us to obtain (8) from the action (7). Typically, in Kaluza-Klein theories, one assumes the compactification $M^{d+1} \rightarrow M^{3+1} \times B$, where M^{3+1} is identified with the ordinary four dimensional manifold and B is considered as $(d-3)$ -dimensional compact manifold. For instance, in $D=11$ supergravity one may consider the compactification $M^{10+1} \rightarrow M^{3+1} \times S^7$, where S^7 is the seven sphere. It is assumed that the "size" of the compact manifold B , with isometry group G , is much smaller than the "size" of the other physical manifold M^{3+1} . This assumptions allow to apply the so called dimensional reduction mechanism. In going from (7) to (8) we have applied similar procedure considering M^{0+8} as a kind of compact manifold and in this way eliminating possible cross terms. However, just as the Kaluza-Klein theory has several interesting generalizations, including arbitrary B manifolds (see, for instance, Ref. [33]), for further research it may be interesting to investigate a possible interaction between the two sectors M^{2+2} and M^{0+8} .

Finally, it is worth mentioning that, besides the $N=(2,1)$ heterotic string theory, Matrix theory [34] is another proposed candidate for M -theory. Since there seems to exist a connection between these two approaches (see Ref. [35]) one should also expect a relation between Ashtekar formalism and Matrix theory. Fortunately, Smolin [36]-[37] (see also Refs [38] and [39]) already has described this possibility. In particular, in the context of topological M -theory

Smolin [37] has investigated the possibility of obtaining Hitchin's 7 seven dimensional theory, which in principle seems to admit background independent formulation, from the classical limit of M -theory, namely eleven dimensional supergravity. The idea is focused on an attempt of reducing the eleven dimensional manifold M^{1+10} in the form

$$M^{1+10} \rightarrow R \times \Sigma \times S^1 \times R^3. \quad (54)$$

Here, Σ is a complex six-dimensional manifold. Considering that the only degree of freedom is the gauge field three form A which is pure gauge $A = d\beta$ and therefore locally trivial $dA = 0$, the Smolin's conjecture is that the Hitchin's action can be derived from the lowest dimensional term that can be made from $d\beta$ on $R \times \Sigma$ of the corresponding effective action (see Ref. [37] for details). Observing that $\Sigma \times S^1$ is a seven dimensional manifold and since, via the octonion structure, the solution $0 + 8$ is related to the seven sphere solution of eleven dimensional supergravity one is motivated to conjecture that there must be a connection between our approach of incorporating Ashtekar formalism in the context of M -theory and the Smolin's program.

References

- [1] I. Bars, *Twistor and 2t-physics*, AIP Conf. Proc. (Wroclaw 2004, Fundamental interactions and twistor-like methods) 767, 3-27 (2005); hep-th/0502065.
- [2] H. Nishino, Phys. Lett. B **428**, 85-94 (1998); hep-th/9703214.
- [3] S. Hewson and M. Perry, Nucl. Phys. B **492**, 249 (1997); hep-th/9612008
- [4] Ooguri and C. Vafa, Nucl. Phys. B **367**, 83 (1991); Nucl. Phys. B **361**, 469 (1991).
- [5] S. V. Ketov, H. Nishino and S. J. Gates, Phys. Lett. B **307**, 323 (1993); hep-th/9203081.
- [6] J. A. Nieto, Nuovo Cim. B **120**, 135 (2005); hep-th/0410003.
- [7] H. Nishino and S. Rajpoot, JHEP **0404**, 020 (2004); hep-th/0210132; Phys. Lett. B **564**, 269 (2003); hep-th/0302059.
- [8] J. A. Nieto and L. N. Alejo-Armenta, Int. J. Mod. Phys. A **16**, 4207 (2001); hep-th/0005184.

- [9] A. Ashtekar and J. Lewandowski, *Class. Quant. Grav.* **21**, R53 (2004); gr-qc/0404018.
- [10] J. Zanelli, M. Henneaux and C. Teitelboim, *Gravity in Higher Dimensions*, Rio de Janeiro Proceedings 1987, SILARG VI, 70-95.
- [11] S. W. MacDowell and F. Mansouri, *Phys. Rev. Lett.* **38**, 793 (1977); F. Mansouri, *Phys. Rev. D* **16**, 2456 (1977).
- [12] P. G. O. Freund, *Introduction to Supersymmetry* (C. U. P. 1986).
- [13] J. A. Nieto, O. Obregón and J. Socorro, *Phys. Rev. D* **50**, R3583 (1994); gr-qc/9402029.
- [14] J. A. Nieto, *Class. Quant. Grav.* **22**, 947 (2005); hep-th/0410260.
- [15] J. A. Nieto, J. Socorro and O. Obregón, *Phys. Rev. Lett.* **76**, 3482 (1996).
- [16] J. A. Nieto, *Mod. Phys. Lett. A* **20**, 2157 (2005), hep-th/0411124.
- [17] M. Gunaydin and Gursev, *J. Math. Phys.* **14**, 1651 (1973).
- [18] A. R. Dunderer, F. Gursev and C. H. Tze, *J. Math. Phys.* **25**, 1496 (1984).
- [19] A. R. Dunderer and F. Gursev, *J. Math. Phys.* **32**, 1178 (1991).
- [20] K. Sfetsos, *Nucl. Phys. B* **629**, 417 (2002); hep-th/0112117.
- [21] D. Mülsch and B. Geyer, *Int. J. Geom. Meth. Mod. Phys.* **1**, 185 (2004); hep-th/0310237.
- [22] D. Kutasov and E. J. Martinec, *Nucl. Phys. B* **477**, 652 (1996), hep-th/9602049; *Class. Quant. Grav.* **14**, 2483 (1997), hep-th/9612102.
- [23] A. Björner, M. Las Vergnas, B. Sturmfels, N. White and G. M. Ziegler, *Oriented Matroids*, (Cambridge University Press, Cambridge, 1993)
- [24] J. A. Nieto, *Adv. Theor. Math. Phys.* **8**, 177 (2004); hep-th/0310071
- [25] J. A. Nieto and M.C. Marín, *Int. J. Mod. Phys. A* **18**, 5261 (2003); hep-th/0302193.
- [26] J. A. Nieto and M. C. Marín, *J. Math. Phys.* **41**, 7997 (2000).
- [27] J. A. Nieto, *J. Math. Phys.* **45**, 285 (2004); hep-th/0212100.

- [28] J. A. Nieto, *Rev. Mex. Fis. E* **51**, 5 (2005); hep-th/0407093.
- [29] J. A. Nieto, "Oriented Matroid Theory as a Mathematical Framework for the M-theory", to be published in *Adv. Theor. Math. Phys.*, hep-th/0506106.
- [30] J. A. Nieto, "Toward a Connection Between the Oriented Matroid Theory and Supersymmetry"; hep-th/0510185.
- [31] T. Jacobson and L. Smolin, *Class. Quant. Grav.* **5**, 583 (1988).
- [32] J. Samuel, *Pramana J. Phys.* **28**, L429 (1987).
- [33] Y. M. Cho, K. S. Soh, J. H. Yoon and Q-Han Park, *Phys. Lett. B* **286**, 251 (1992).
- [34] T. Banks, W. Fischler, S. H. Shenker, L. Susskind, *Phys. Rev. D* **55**, 5112 (1997); hep-th/9610043.
- [35] E. J. Martinec, "Matrix theory and N=(2,1) strings" hep-th/9706194.
- [36] L. Smolin, *Phys. Rev. D* **62**, 086001 (2000); hep-th/9903166.
- [37] L. Smolin, *Nucl. Phys. B* **739**, 169 (2006); hep-th/0503140.
- [38] Y. Ling and L. Smolin, *Phys. Rev. D* **61**, 044008 (2000); hep-th/9904016.
- [39] T. Thiemann, *Class. Quant. Grav.* **23**, 1923 (2006); hep-th/0401172.